

Phase diagram of optimal paths

Alex Hansen*

Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

János Kertész†

Institute of Physics, Budapest University of Technology, Budafoki út 8, H-1111 Budapest, Hungary

(Dated: February 2, 2008)

We show that choosing appropriate distributions of the randomness, the search for optimal paths links diverse problems of disordered media like directed percolation, invasion percolation, directed and non-directed spanning polymers. We also introduce a simple and efficient algorithm, which solves the d -dimensional model numerically in $\mathcal{O}(N^{1+d_f/d})$ steps where d_f is the fractal dimension of the path. Using extensive simulations in two dimensions we identify the phase boundaries of the directed polymer universality class. A new strong-disorder phase occurs where the optimum paths are self-affine with parameter-dependent scaling exponents. Furthermore, the phase diagram contains directed and non-directed percolation as well as the directed random walk models at specific points and lines.

PACS numbers: 05.40.+j, 02.50.-r, 47.55.Mb, 64.60.Ak

The search for optimal paths in a random environment is one of the basic problems of statistical physics [1]. The task is to find the path between two points in a random energy landscape such that the total energy along that path is minimal. After a mapping to a graph where the edges bear random weights we arrive at a widely studied problem of discrete mathematics (see, e.g., [2]). From the point of view of physics, the interest comes from intimate relations to various fields including the geometry of domain walls in disordered magnets [3, 4, 5], vortices in superconductors [6] or rupture lines [7]. For d -dimensional *directed* lines without overhangs, often referred to as “directed polymers in random media” (DPRM), there is an exact mapping to stochastic surface growth [8], enabling further connections to fractal surfaces [9], to driven particle systems [10] and to the stochastic Burgers equation [11]. The main questions for all these applications are: How can the geometry of the optimal paths be described and how do the energy fluctuations scale with their length? Some realizations of optimal paths for different disorders are shown in Fig. 1.

The situation is quite clear for the DPRMs: The line is a self-affine fractal resulting in scaling endpoint-fluctuations. Imagine that the polymer is held fixed at one end and we ask for conformations of length L over different disorder realizations. Let $(\Delta x)^2$ be the average squared endpoint-fluctuations of the polymer, then $\Delta x \propto L^\zeta$ where ζ is the roughness exponent. Similarly, for the average squared energy fluctuations $(\Delta E)^2$ we have $\Delta E \propto L^\omega$. In $d = 2$, the exponents are known exactly [4]: $\zeta_{DPRM} = 2/3$ and $\omega = 1/3$. In higher dimensions numerically calculated values are available [12].

Recently, the question of the conformation of *span-*

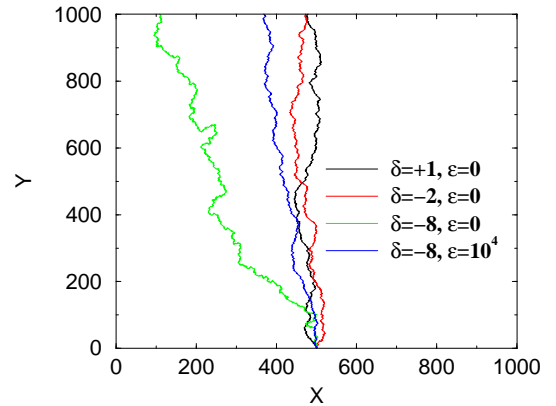


FIG. 1: Optimal paths for different values of the disorder parameters δ and ϵ , defined in Eq. (3).

ning non-directed polymers has attracted considerable interest. Indeed, the original optimal path problem does not exclude any geometry a priori, in principle overhangs should be allowed. In fact, it can be easily shown that overhangs have to occur on a sufficiently long line. However, recent numerical work on such non-directed polymers in a random medium (NDPRMs) have given evidence that, for bounded unimodal distributions of disorder, these overhangs are irrelevant [13] and the NDPRMs are in the universality class of DPRMs [14, 15]. The latter conclusion was also drawn from a real space renormalization group study [13].

Most of the studies of optimum paths have been confined to Gaussian distribution of the randomness. The effect of changing the distribution has been studied in some recent papers. In spite of earlier reports [21], universality of the DPRMs was found for wide family of bounded distributions [22]. It was shown [19] that, when the distribution has a power law decaying part for negative energies characterized by an exponent μ , the effect

*Alex.Hansen@phys.ntnu.no

†Kertesz@phy.bme.hu

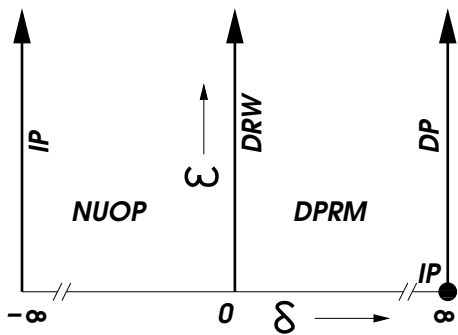


FIG. 2: Phase diagram in the disorder parameter δ and the directedness parameter ε , defined in Eq. (3). DPRM: Directed Polymer in a Random Medium, IP: Invasion Percolation, DP: Directed Percolation, NUOP: Non-Universal Optimal Paths.

described by Zhang [20] for surface growth can be observed for DPRMs too: The scaling exponents of the paths depend continuously on μ in an interval of μ . For a bimodal distribution, where the energy can be either 1 or 0 (with probability p and $1-p$, respectively) there are crossover phenomena near to the percolation threshold p_c [13, 23]. Long range correlations in the randomness may lead to nonuniversal behavior [24].

The standard models of disorder, directed percolation (DP) or ordinary percolation [25] can also be formulated in terms of global optimization problems [16], as long as the spanning paths are concerned. We briefly summarize these cases [17]. The optimal path \mathcal{P} in the polymer problem is given by

$$\min_{\mathcal{P}} \sum_{i \in \mathcal{P}} E_i, \quad (1)$$

while in the percolation problems we have for the spanning paths \mathcal{P}

$$\min_{\mathcal{P}} \left(\max_{i \in \mathcal{P}} \right) E_i, \quad (2)$$

where for the directed versions the condition of directedness to (1) and (2) have to be superimposed. The only difference between these equations is that in (2) we have the maximum instead of the sum in (1). When looking for the bottlenecks along each portion of the path we obtain exactly an optimal percolation path. However, this path does not scale like the shortest (or chemical) distance on a critical percolation cluster, but rather as one on an invasion percolation (IP) tree [5, 26], or trapping invasion percolation [26]. These invasion percolation problems have much in common with ordinary percolation, e.g., the critical point and the fractal dimension of the critical cluster agree.

The geometry of the optimal percolation paths is also known. For DP we have again a self-affine line with $\zeta_{DP} = 0.633$ in two dimensions [18], calculated from the ratio of the perpendicular and the parallel correlation

length exponents. For IP we have a fractal line with a dimension $d_f > 1$. In two dimensions its value is $d_f = 1.22$ [5, 26]. Naturally, the roughness exponent for a fractal line is $\zeta_{IP} = 1$.

It is tempting to try to formulate an unified picture of these diverse but obviously related phenomena and in this Letter we address this point. Furthermore we would like to explore the possibility of new universality classes and at the same time to check earlier results by improving the numerics.

Let us look for a parametrized distribution of randomness such that both directedness and strong disorder naturally occur as limits. In order to achieve this, weights are assigned to each bond as

$$E = r^\delta + \varepsilon, \quad (3)$$

where $r \in (0, 1)$ is taken from an uniform random distribution; $\varepsilon \geq 0$ and $-\infty \leq \delta \leq \infty$ are the directedness and the disorder parameters, respectively. This choice corresponds to a PDF in the reduced variable $\tilde{E} = E - \varepsilon$:

$$p(\tilde{E}) = \frac{\tilde{E}^{1/\delta-1}}{|\delta|}. \quad (4)$$

Here $\tilde{E} \in (0, 1]$ for $\delta > 0$ while $\tilde{E} \geq 1$ for $\delta < 0$. The optimal path is searched for in a random environment characterized by the bond weights in (3), resulting in complex phase diagram, Fig. 2.

Already a simple analysis of (3) reveals interesting connections to different models of statistical physics. Let us take first $\delta = 0$. Along this line a penalty is to be paid for any overhang. However, all directed paths have the same weights. This is exactly the statistics of directed random walks (DRW), leading to self affine lines with roughness exponent $\zeta_{DRW} = 1/2$ [27].

The case of NDPRM with a bounded distribution can be identified with $\varepsilon = 0$, $0 < \delta \leq \infty$. The optimal paths are in the universality class of DPRM [14, 15]. Obviously, this is even more so when ε , the parameter suppressing overhangs, is switched in. Interestingly, no crossover phenomena at $\delta \rightarrow 0$ are to be expected since however small δ is the global optimization over the random weights matters, since the ε weights represent only a constant background.

The limit $\delta \rightarrow +\infty$ the distribution is more and more shifted towards a delta function at 0. Nevertheless, this limit corresponds to *strong disorder* as it can be shown with reference to the order statistics [28]: The cumulative distribution is $P(E) = \int_0^E p(E') dE' = E^{1/\delta}$ (we first assume $\varepsilon = 0$). We generate an ensemble of groups of E -values, N in each group. We order the numbers in each group in an ascending order so that $E_{(1)} \leq E_{(2)} \leq \dots \leq E_{(N)}$. The average value of the n th element in this ordering is $\langle E_{(n)} \rangle = [n/(N+1)]^\delta$. Hence, the ratio between consecutive elements in this ordering is given by $\langle E_{(n+1)} \rangle / \langle E_{(n)} \rangle = (1 + 1/n)^\delta$ which diverges for any n as $\delta \rightarrow +\infty$. Consequently, any sum of elements drawn from this distribution will be dominated by the largest

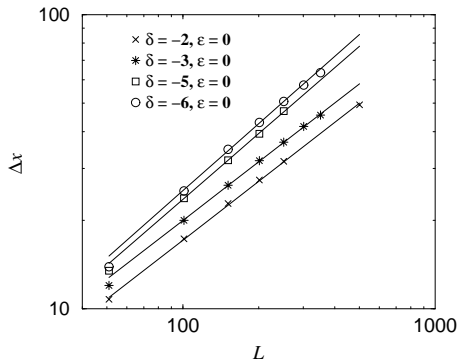


FIG. 3: Δx as a function of system size L . The slopes of the $(\delta = -2, \varepsilon = 0)$ and $(-3, 0)$ lines are 0.67, while the slopes of the $(\delta = -5, \varepsilon = 0)$ and $(-6, 0)$ lines are 0.75.

element, showing that we are in the infinite disorder limit, and the optimum paths are percolation paths [16]. For $\varepsilon = 0$ it is the invasion percolation shortest path [26]. However small value of ε is taken it will immediately suppress the overhangs transforming the optimal path to a critical *directed* percolation path.

In the regime $\delta < 0$ the distribution has a power law tail with the consequence that only the $k < 1/|\delta|$ -th moments remain finite. If this property is inherited to the optimal paths, this would mean that there should be nonuniversal behavior as a function of δ since the universality classes are characterized not only by the exponents but also by the universal scaling functions. Moreover, the large number of huge obstacles generated by the tail of the distribution may lead to an enhancement of overhangs resulting in new geometries. We apply numerical tools in order to see clearly in this point.

Before that, let us summarize the behavior at $\delta \rightarrow -\infty$. In this limit, the distribution becomes not normalizable indicating the dominance of large values of E , i.e., this corresponds again to the infinite disorder limit, where the arguments of order statistics apply straightforwardly and we arrive at (2) which is the percolation case. It is a question of interest whether the parameter ε is relevant for this limit and we will come later back to this point.

Let us turn to our two-dimensional numerical simulations. The obvious difficulty in the determination of optimal paths is in the global nature of the problem. Given an optimal path of span L , the solution for size $L + 1$ can be entirely different. However, there are algorithms which calculate the optimal path on graph with random positive weights on the edges in polynomial time [2]. We have developed an efficient algorithm which is adequate to the geometry we considered.

We now describe this algorithm in some detail. It is closely related to the algorithm described by Hansen and Hinrichsen in [16] for determining the non-directed percolation threshold. We describe the algorithm for the two-dimensional case, i.e., for a 45° tilted square lattice. The sample has a cylindrical geometry (periodic bound-

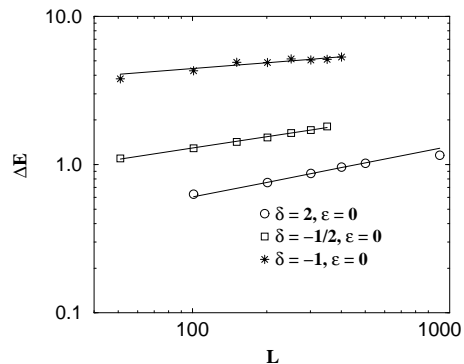


FIG. 4: ΔE as a function of system size L . The slopes are for $(\delta = 2, \varepsilon = 0)$ 0.33, for $(-1/2, 0)$, 0.25 and for $(-1, 0)$ 0.13.

ary conditions in the horizontal direction). Each bond i is assigned an energy E_i . We associate a variable V_α with each node α . Initially, these variables are all set to zero with the exception of the nodes forming the horizontal edges of the lattice. These nodes are assigned a value which is larger than any energy E_i appearing in the system. Furthermore, the node forming the anchor-point of the polymer, $\alpha = 0$ is assigned a negative value $V_0 = v_0 = -|v_0|$. This node sits on the lower edge of the lattice. We then iteratively update the *internal* vertices (i.e. those not being on the horizontal edges of the lattice) according to the scheme

$$V_\alpha \rightarrow \min(V_{\alpha(1)} + E_{\alpha(1)}, V_{\alpha(2)} + E_{\alpha(2)}, V_{\alpha(3)} + E_{\alpha(3)}, V_{\alpha(4)} + E_{\alpha(4)}),$$

where $\alpha(1)$ to $\alpha(4)$ are the four nodes that neighbor node α and the four bonds joining them. The updating proceeds until the node variables no longer change. The value of each node variable is then equal to the energy of the optimal path between that particular node and the anchor node plus the value v_0 . The number of iterations goes as $L \times$ the length of the shortest path, i.e., as L^{1+d_f} , where d_f is the fractal dimension of that path.

In Fig. 3 we show end point fluctuations Δx as a function of system size L for a number of different disorders. For $\delta \geq -2$ and $\varepsilon = 0$, we find $\Delta x \sim L^{0.67}$, indicating self affinity with the same roughness exponent as in the DPRM problem, where $\zeta_{DPRM} = 2/3$. We also find the same behavior with $\delta = -8$ and $\varepsilon = 10^4$. However, for disorders with $\delta < -5$ or less and $\varepsilon = 0$, we find $\Delta x \sim L^{0.75}$, which indicates self affine behavior, but with a new roughness exponent which is significantly larger than the DPRM one. The self affinity of the optimal paths is further supported by monitoring the average length of the paths, l as a function of system size L . For all finite disorders investigated, we find $l \sim L^{1.00}$. Hence, the curve is *not* a self-similar fractal, and the overhangs are irrelevant. Fig. 1 shows different optimal paths with different disorders. Even in the most extreme cases ($\delta = -8$ and $\varepsilon = 0$), there are few overhangs.

The fluctuations in energy, ΔE , scales as $\Delta E \sim L^{1/3}$

in the DPRM case. For $\delta > -1$ we find that ΔE is self averaging. In Fig. 4, we show ΔE as a function of L for $\delta = 2$, $\delta = -1/2$, and $\delta = -1$, all with $\varepsilon = 0$. For $\delta = 2$ case, we find an exponent consistent with the DPRM value, $1/3$. However, for the $\delta = -1/2$, we find a value which is much smaller, 0.25 , while for $\delta = 1$, we find 0.13 . This may be a crossover effect.

For values of $\delta < -1$, ΔE is not self averaging, indicating no universal scaling functions exist. However, the scaling of Δx with two different values of scaling exponent suggests that there are still partial universality in the system for small values of δ and $\varepsilon = 0$, with a roughness exponent of 0.75 .

Universality classes are characterized by scaling functions, not only by exponents [29]. Therefore we extended our simulations to higher moments of the endpoint fluctuations (wherever it was possible), because the amplitude ratios are known with high precision [29] and also to the study of the moments of the energy fluctuations. From our studies we conclude: i) For $0 < \delta < \infty$ the amplitude ratios are within the numerical accuracy the same as for

DPRM, confirming universality. ii) For $0 > \delta > -\infty$ the k -th moments of the energy fluctuations become divergent for $k > 1/|\delta|$ (i.e., even the second moment is nonexistent for $\delta < 2$) indicating *nonuniversal* behavior (see. Fig. 2: Non-universal optimal paths, NUOP). The measurements were carried out for $\varepsilon = 0$. In the case of $\varepsilon > 0$ the situation is even worse: While ε does not contribute to the energy fluctuations, its effect is enhanced directedness, i.e., the path is forced through higher E valued regions. However, it seems that universality in a restricted sense is present: The roughness exponent seems to agree with ζ_{DPRM} for $\delta \geq -4$ and $\varepsilon = 0$. Also for large values of ε the ζ becomes close to $2/3$ even for large negative values of δ . Furthermore, the roughness exponent for $\delta < -4$ is ≈ 0.75 . Further investigations are needed to explore fully this restricted universality.

We thank K. Sneppen and S. Havlin for useful comments. Support was provided by the NFR through grant SUP/10225700. JK is member of the Center for Applied Mathematics and Computational Physics at the BUTE.

-
- [1] T. Halpin-Healy, Y.-C. Zhang, Physics Reports **254**, 215 (1995); H. Rieger, M. Alava, P. Duxbury and C. Moukarzel in: *Phase Transitions and Critical Phenomena*, Eds. C. Domb and J. Lebowitz, Vol. 18, p. 141, (Cambridge UP, 2000).
 - [2] E.S. Lawler, *Combinatorial Optimization: Networks and Matroids* (Dover, New York, 2001).
 - [3] D.A. Huse and C. Henley, Phys. Rev. Lett., **54**, 2708 (1985).
 - [4] M. Kardar and D.R. Nelson, Phys. Rev. Lett. **55**, 1157 (1985); D.A. Huse, C. Henley and D.S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).
 - [5] M. Cieplak, A. Maritan and J. Banavar, Phys. Rev. Lett. **72**, 2320 (1994).
 - [6] D.R. Nelson, Phys. Rev. Lett. **60**, 1963 (1988); M. Kardar, Phys. Rep. **301**, 85 (1998).
 - [7] K.J. Måløy, A. Hansen, E.L. Hinrichsen, and S. Roux, Phys. Rev. Lett. **68**, 213 (1992); J. Kertész, V.K. Horváth, and F. Weber, Fractals **1**, 67 (1993).
 - [8] M. Kardar, G. Parisi and Y.-C. Zhang, Phys. Rev. Lett., **56**, 889 (1986).
 - [9] See Halpin-Healy and Zhang in [1], F. Family and T. Vicsek (eds.) *Dynamics of Fractal Surfaces* (World Sci., Singapore, 1993); A.-L. Barabási and H.E. Stanley: *Fractal Surface Growth* (Cambridge Univ. Press, Cambridge, 1995).
 - [10] B. Derrida, Phys. Rep. **301**, 65 (1998).
 - [11] W. E, K. Khanin, A. Mazel and Y. Sinai Phys. Rev. Lett. **78**, 1904 (1997).
 - [12] B. Forest and L.-H. Tang, J. Stat. Phys. **60**, 181 (1990); E. Marinari, A. Pagnani, G. Parisi, J. Phys. A, **33**, 81881 (2000).
 - [13] F. Seno, A.L. Stella, C. Vanderzande, Phys. Rev. E, **55**, 3859 (1997),
 - [14] M. Marsili and Y.C. Zhang, Phys. Rev. E **57**, 4814 (1998); N. Schwartz, A. L. Nazaryev, S. Havlin, Phys. Rev. E **58**, 7642 (1998).
 - [15] R. Schorr and H. Rieger, Eur. Phys. J. B **33**, 347 (2003).
 - [16] This relationship was first pointed out in S. Roux, A. Hansen and E. Guyon, J. Phys. (Paris), **48**, 2125 (1987) for directed percolation and in A. Hansen and E.L. Hinrichsen, Phys. Script. T **44**, 55 (1992) for non-directed percolation. In these papers algorithms are presented.
 - [17] We consider a hypercubic lattice limited by two planes at distance L , such that the line perpendicular to the planes is parallel to the diagonals of the elementary hypercubes.
 - [18] A.J. Guttmann and K. De'Bell, J. Phys. A **21**, 3815 (1988).
 - [19] S. Roux, A. Hansen and E.L. Hinrichsen, J. Phys. A **24**, L295 (1991); L.-H. Tang, J. Kertész and D.E. Wolf, J. Phys. A **24**, L1193 (1991).
 - [20] Y.-C. Zhang, J. Phys. (Paris) **51**, 2129 (1990).
 - [21] T.J. Newman and M.R. Swift, Phys. Rev. Lett. **79**, 2261 (1997).
 - [22] E. Perlsman and S. Havlin, Phys. Rev. E, **63**, 010102 (2000).
 - [23] T. Halpin-Healy, Phys. Rev. E, **58**, R4096 (1998). E. Perlsman, S. Havlin, Europhys. Lett. **46**, 13 (1999). J.S. Andrade et al. Phys. Rev. **62**, 8270 (2000).
 - [24] Y.-C. Zhang, Phys. Rev. Lett. **56**, 2113 (1986); E. Medina, T. Hwa, M. Kardar and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).
 - [25] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1994); Fractals and Disordered Systems, edited by A. Bunde and S. Havlin, eds., *Fractals and Disordered Systems* (Springer, Berlin, 1994).
 - [26] M. Porto, S. Havlin, S. Schwartz and A. Bunde Phys. Rev. Lett., **79**, 4060 (1997).
 - [27] B.K. Chakrabarti and S.S. Manna, J. Phys. A **16**, L113 (1983).
 - [28] H.A. David, *Order Statistics* (Wiley, New York, 1981).
 - [29] T. Halpin-Healy, Phys. Rev. A **44**, R3415 (1991); A. Hansen and J. Kertész, Phys. Rev. E **53**, R5541 (1996).